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SYSTEMS CONTROL, INC.
1801 PAGE MILL ROAD
PALO ALTO, CALIFORNIA 94304

TELEX: 348433

TELEPHONE (415)
494-1165

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NETWORKS OF QUEUES - EQUILIBRIUM ANALYSIS

by

Austin J. Lemoine

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Approved by:

E. Arlin Torbett

E. Arlin Torbett, Project Director
Manager Telecommunication Sciences Program
Systems Control, Inc.

CONTROL ANALYSIS CORPORATION
800 Welch Road
Palo Alto, California 94304

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Queuing network models abound in applications but, despite their importance, many of the results for these models (both classical and recent) are not well known. The intent of this paper is to provide an overview of available results on the equilibrium analysis of networks of queues, along with the methodology which has been employed to obtain those results. In addition, we call attention to some important open problems. Detailed discussion is limited to a small group of papers which seem important to us, and there is -> next page

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ABSTRACT

Queueing network models abound in applications, but despite their importance many of the results for these models, both classical and recent, are not well known. The intent of this paper is to provide an overview of available results on the equilibrium analysis of networks of queues, along with the methodology which has been employed to obtain those results. In addition, we call attention to some important open problems. Detailed discussion is limited to a small group of papers which seem important to us, and there is considerable emphasis on recent work appearing in the applied probability literature. The list of references does include, however, work not reviewed here, along with other survey and background material.

1.0 INTRODUCTION

Queueing network models abound in applications, appearing in such important and diverse areas as communications networks and teletraffic, computer time-sharing and multiprogramming systems, maintenance and repair facilities, production, assembly and inspection operations, airtraffic control, and medical care delivery systems. Despite the importance of queueing network models, however, many of the existing results for these models, both classical and recent, are not well known. In this paper we attempt to provide an overview of the available results on the equilibrium analysis of networks of queues and of the methods which have been used to obtain those results. In addition, we call attention to some interesting open problems. Our discussion is confined to a rather small group of papers which seem important to us, and there is considerable emphasis on recent work which has appeared in the applied probability literature. The list of references does include, however, work not reviewed here, along with other survey and background material.

The work we discuss is devoted, almost exclusively, to finding the equilibrium (or limiting) distribution for a network of queues which can be represented as a continuous time Markov chain with a countable state space (cf. [6] or [7]). The results which have been derived in this area are primarily concerned with the joint equilibrium distribution of the customer queue sizes at the various nodes (or service centers) and with the nature of the processes of customer departures from the network in equilibrium. For a network of queues with N nodes and M types (or classes) customers, the equilibrium state probabilities (when they exist) have the general form (cf. [2] and [15])

$$\pi(C) = B \cdot \gamma(C) \cdot \psi_1(c_1) \cdot \psi_2(c_2) \; . \; . \; . \; \psi_N(c_N) \tag{1}$$

where $C = (c_1, c_2, \ldots c_N)$ and c_i is the configuration of customers at node i, where $\gamma(\cdot)$ is a function of the number of customers in the system, $\psi_i(\cdot)$ is a function that depends on the nature of node i, and B is a normalizing constant. In the simplest case, the configuration c_i may represent the total number of customers at node i; but, in other instances it may also involve the numbers of customers of different types, as well as their arrangement within the queue, at node i. For open networks, that is, networks of queues in which customers originate from external sources and each customer eventually leaves the system, the equilibrium state probabilities often factor into terms in which each term involves only the parameters for a single node; in such cases, for example, the numbers of customers at the various nodes are independent random variables. Also, for most open networks of interest, the process of customer departures from the system consists of independent Poisson streams, be they departures of customer types or departures from the various nodes.

This paper is organized as follows. In Section 2 we present a basic model introduced in the classical paper of Jackson (1957). This basic model is used to provide insight into problem formulation, into standard results for equilibrium distributions and equilibrium departure processes, into the important technique of partial balance for deriving equilibrium distributions, and into the important technique of time reversal for studying departure processes in equilibrium. In Section 2, other classical work following the paper

of Jackson (1957) is also discussed, including the papers of Jackson (1963), Gordon and Newell (1967), and Posner and Bernholtz (1968). The section is closed with a review of some of the limitations of this classical work. In Section 3 we discuss the careful study of the classical model of Jackson (1957) recently put forth by Beutler et al ([3], [4], [30], [31]). Using results from [3], we derive the Laplace transform of the distribution of the total time spent in the system by an arbitrary customer under equilibrium conditions for the important case of an open "acyclic network," that is, an open network in which each customer can visit any node only once. In Section 3 we also discuss the recent generalizations of the models of Jackson (1957) and Gordon and Newell (1967) which have been developed in Kelly (1975), Baskett et al (1975), and Reiser and Kobayashi (1975). In Section 4 we discuss the papers of Kelly (1976) and Barbour (1976) which expand the basic models of Jackson (1957, 1963) particularly with regard to routing behavior. Finally, in Section 5 we discuss some important open problems in the equilibrium analysis of networks of queues.

Other surveys on the subject of networks of queues, and the related areas of computer time-sharing and multiprogramming systems and computer-communications networks, include Disney (1975), Wyszewianski and Disney (1975), Kobayashi (1976), and Kobayashi and Konheim (1977). An introduction to the subject of networks of queues is provided by Section 4.8 of Kleinrock (1975) and Section 4.12 of Kleinrock (1976). Also, Chapters 4 and 5 of Kleinrock (1976) provide comprehensive discussions on the subjects of computer time-sharing and multiaccess systems and of computer-communications networks.

2.0 THE BASIC MODEL

The model introduced in Jackson (1957) is a generalization of the classical $M/M/s$ queue to an arbitrarily interconnected open network of server nodes with exponential service and Poisson external input. Specifically, there are N nodes with node i having s_i servers, a first-come-first-served queue discipline, and a waiting room of unlimited capacity. The external input stream to node i is Poisson with rate λ_i , and these external input streams are assumed to be independent. The service times at node i are independent and have a common exponential distribution with parameter μ_i , and are also independent of the customer arrivals at node i . A customer leaving node i is immediately and independently routed to node j with probability p_{ij} ; and the customer departs the system from node i with probability $q_i = \sum_{j=1}^{N} p_{ij}$. Observe that this model is quite general, encompassing such systems as a finite number of $M/M/s$ queues in tandem, an acyclic network of $M/M/s$ queues, or a network of $M/M/s$ queues with feedback.

In Jackson (1957) the state of the network at time t is taken to be

$$C(t) = \big(c_1(t), c_2(t), \ldots, c_N(t)\big)$$

where $c_i(t)$ is the number of customers at node i at time t , including any customers in service. The state space $\mathscr{C}$ of $\{C(t), t \geq 0\}$ is the countably infinite set of all N-tuples of non-negative integers. Given the independent Poisson external input streams, the exponential service times at

the various nodes, and the independent routing scheme, it follows readily that $\{C(t), t \geq 0\}$ is a Markov process with stationary transition probabilities.

In [14] Jackson proved the remarkable result that whenever an equilibrium condition exists, each node in the network behaves as if it were an independent M/M/s queue with Poisson input. Specifically let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N)$ be the row vector of external input intensities, let P be the $N \times N$ matrix of the p_{ij}'s, and let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N)$ be the row vector solution of the traffic equation (cf. [3])

$$\alpha = \lambda + \alpha P \quad . \tag{2}$$

Since we are assuming the network is open, that is, any customer in any queue eventually leaves the system, it follows that each entry of the matrix P^n converges to 0 as $n \to \infty$. Thus, the matrix $I - P$ is invertible, and the equation (2) has a unique solution for a given λ. Intuitively, the traffic equation is a balance or conservation equation in which α_i is interpreted as the total average arrival rate to node i under equilibrium conditions. And, it is plausible that equilibrium conditions will obtain if the "traffic intensity" is less than one at each node, i.e.,

$$\rho_i \equiv \alpha_i / s_i \mu_i < 1 \tag{3}$$

for $i = 1, 2, \ldots, N$. Under these assumptions, Jackson showed that if $\pi(C)$ is the equilibrium probability of the network being in state $C = (c_1, c_2, \ldots c_N)$, then

$$\pi(C) = \psi_1(c_1) \cdot \psi_2(c_2) \cdots \psi_N(c_N) \tag{4}$$

where $\psi_i(c_i)$ is the equilibrium probability of having c_i customers in an $M/M/s_i$ queue with input rate α_i and service rate μ_i for each server. The particular product form of the right side of (4) reveals the independence of the various nodes in the network under equilibrium conditions. That is to say, if $C(0)$ is given the distribution (4), then $C(t)$ has distribution (4) for each $t \geq 0$, and so the numbers of customers at the various nodes at each time point are independent random variables. In Figure 1 below we depict the structure of a typical node, say node i, under equilibrium conditions.

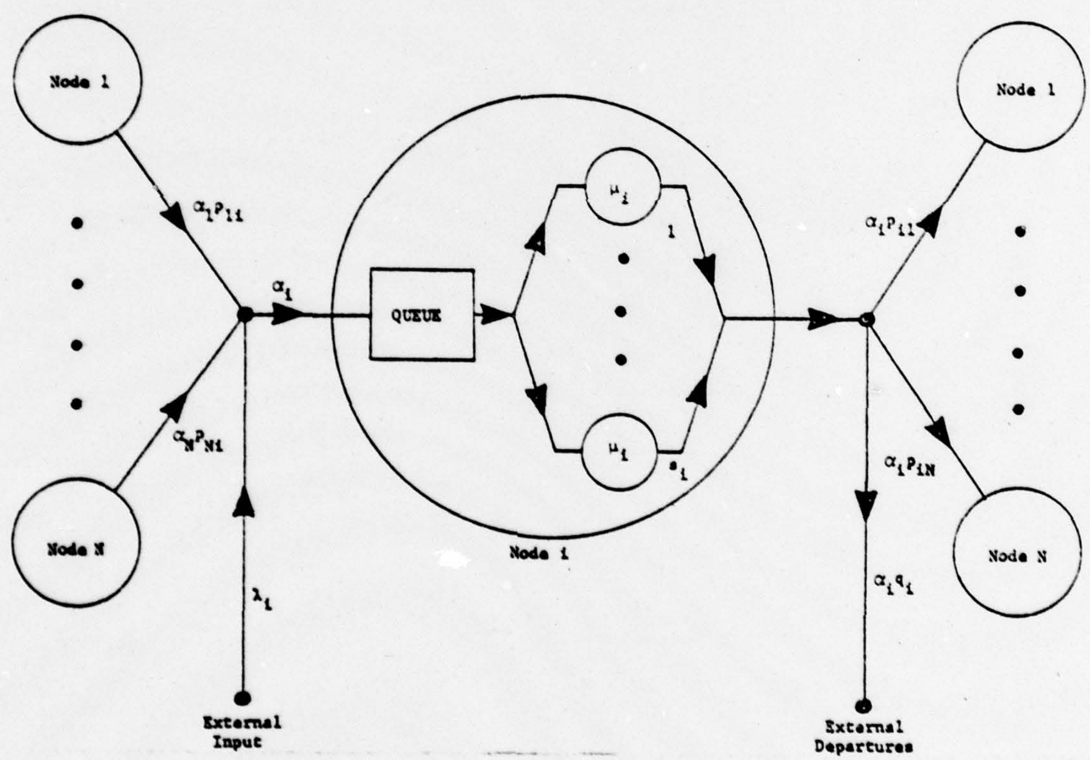

Figure 1--Node i in Equilibrium.

From Figure 1 we see that

$$\sum_{i=1}^{N} \lambda_i = \sum_{i=1}^{N} \alpha_i q_i \quad , \tag{5}$$

that is, under equilibrium conditions, the total external input flow rate to the network must equal the total external departure flow rate. The equation (5) can be readily obtained from the traffic equation (2).

We now proceed to outline a proof of (4). In so doing, we employ an argument which uses the technique of partial balance and which is fundamental to much of the equilibrium analysis of networks of queues. To keep notational complexity to a minimum, <u>we will assume in the upcoming sub-section that each node consists of a single server</u>. Then, in the notation of (4), $\psi_i(c_i) = (1 - \rho_i)\rho_i^{c_i}$ for $i = 1, 2, \ldots, N$.

2.1 The Equilibrium Distribution and Partial Balance

As will be observed later, the continuous time Markov chain $\{C(t), t \geq 0\}$ is irreducible and "well-behaved." Let $q(C)$ denote the total transition rate out of state C and $q(C, D)$ the transition rate from state C to state D for $D \neq C$. In view of (3), for $\pi(C)$ defined by (4) we have

$$\sum_{\mathscr{C}} \pi(C) = 1 \quad . \tag{6}$$

Thus $\{\pi(C), C \in \mathscr{C}\}$ is the equilibrium or limiting distribution of $\{C(t), t \geq 0\}$ if and only if

$$\pi(C)\,q(C) = \sum_{D \neq C} \pi(D)\,q(D,C) \qquad (7)$$

for all states C in $\mathscr{C}$. The equation (7) is a "balance relation" equating the equilibrium rate of flow out of state C with the equilibrium rate of flow into state C. Granting that one has been sufficiently astute to guess the correct form of the equilibrium probabilities, verifying that the probabilities in (4) do satisfy (7) is not entirely trivial. Moreover, as model complexity increases, the direct verification that a given set of probabilities satisfies the balance equation (7) becomes extremely difficult. Fortunately, however, the method of partial balance introduced in Whittle (1967, 1968) provides a simple means for getting around this impasse in many (but not all) cases. In this approach, one attempts to decompose the balance equation (7) into smaller sets of "partial balance equations" and then show that the distribution $\{\pi(C),\ C \in \mathscr{C}\}$ satisfies these simpler equations. For the first such equation we might balance the rate at which the system leaves state C due to external arrivals with the rate at which the system enters state C due to external departures. For the second such equation, we might balance the rate of system transition out of state C due to service completion with the rate of transition into state C due to external arrivals or internal transfers. We might also try to decompose one or both of the above partial balance equations into yet simpler balance relations. For example, with regard to the second of the proposed partial balance equations, we might, for each node i, equate the rate of flow out of state C due to the departure of a customer from node i with the rate of flow into the state C due to the arrival of a customer at

node i . It is important to note that for any given model one has <u>no assurance `a priori</u> that a set of partial balance equations is consistent; cf. [2]. But, it is clear that any solution of a set of partial balance equations will indeed satisfy (7).

To develop partial balance equations for the model of Jackson (1957) with single-server nodes we introduce some additional notation. For $1 \leq i \leq N$ let E_i denote the N-vector with all components zero except for 1 in component i . If $C(t) = C$, then the next transition will be either to state $C + E_i$ (external arrival at node i), to state $C - E_i$ (external departure from node i), or to state $C - E_i + E_j$ for $i \neq j$ (transfer from node i to node j) . Thus, for state C we should have

$$\begin{aligned} q(C) &= \sum_{i=1}^{N} q(C, C+E_i) + \sum_{i=1}^{N} q(C, C-E_i) \\ &\quad + \sum_{i=1}^{N} \sum_{j=1}^{N} q(C, C-E_i+E_j) \end{aligned} \tag{8}$$

where $q(C,C') = 0$ if C' is not in $\mathscr{C}$. We will assume that equation (8) indeed holds for all states C . Then, the first partial balance equation suggested above is

$$\pi(C)\sum_{i=1}^{N} q(C, C+E_i) = \sum_{i=1}^{N} \pi(C+E_i)\, q(C+E_i, C) \quad ,$$

or

$$\pi(C)\sum_{i=1}^{N} \lambda_i = \sum_{i=1}^{N} \pi(C+E_i)\, q_i \mu_i \quad . \tag{9}$$

The second suggested partial balance equation might, as proposed above, be decomposed on a node-by-node basis into

$$\pi(C)\,q(C, C-E_i) + \pi(C)\sum_{\substack{j=1\\ j\neq i}}^{N} q(C, C+E_j-E_i) =$$

$$\pi(C-E_i, C)\,q(C-E_i, C) + \sum_{\substack{j=1\\ j\neq i}}^{N} \pi(C+E_j-E_i)\,q(C+E_j-E_i, C)$$

for $i = 1, 2, \ldots N$, or

$$\pi(C)\left[q_i\mu_i + \sum_{\substack{j=1\\ j\neq i}}^{N} \mu_i p_{ij}\right] = \pi(C-E_i)\,\lambda_i + \sum_{\substack{j=1\\ j\neq i}}^{N} \pi(C+E_j-E_i)\,p_{ji}\mu_j \tag{10}$$

for $i = 1, 2, \ldots, N$, where, for ease of notation, we have assumed in (10) that each component of C is non-zero. In view of (2), (3) and (5), it is easy to check that $\{\pi(C), C \in \mathscr{C}\}$ as defined by (4) satisfies (9) and (10), and hence satisfies (7). It follows immediately from (8) that

$$0 < \sum_{i=1}^{N} \lambda_i \leq q(C) \leq \sum_{i=1}^{N} \lambda_i + \sum_{i=1}^{N} \mu_i < \infty$$

for all states C. Thus, the process $\{C(t), t \geq 0\}$ has no absorbing states and has only finitely many transitions in any finite time interval with

probability one. Since any customer in any queue eventually leaves the system, then starting from any state in $\mathscr{C}$ it is possible by a series of transitions to reach the state $(0,0,\ldots,0)$. Hence, the process $\{C(t), t \geq 0\}$ is also irreducible. The proof of (4) is now complete.

2.2 Departure Processes and Time Reversal

In addition to the "product form" of the equilibrium queue-lengths distribution, the other main equilibrium result for open Markovian networks of queues is concerned with the process of external departures. As remarked earlier, this equilibrium departure process consists of independent Poisson streams. In order to illustrate the result, let us once again consider the model of Jackson (1957).

By a result of Burke (1956) and Reich (1957), we know that the output of an M/M/s queue in equilibrium is Poisson with the same intensity as the input process. From (4) we know that the queue-length processes at the various nodes are independent in equilibrium, and that node i behaves as a $M/M/s_i$ queue with input rate α_i. Moreover, a customer leaving node i exits the system with probability q_i. Thus, it is plausible that <u>in equilibrium the external departure streams are independent Poisson processes, the rate at node i being</u> $q_i\alpha_i$; cf.[4] or [19]. To verify this we outline an argument using <u>time reversal</u>.

The choice of the origin on the time axis as the starting point for the equilibrium queue-lengths process is entirely arbitrary. If we give $C(0)$ the distribution defined by (4) then the equilibrium process $\{C(t), 0 \leq t < \infty\}$ can be regarded as the continuation of a process in equilibrium that has already

been going on an infinite length of time. To see why this is so, first note that for t' in $(-\infty, 0)$ we can define the equilibrium process over $[t', \infty)$ by simply moving the starting point from 0 back to t'. Letting t' tend to $-\infty$ then gives rise to a consistent family of finite dimensional distributions (cf. [5]) which enables one to construct an equilibrium queue-lengths process defined over all times t in $(-\infty, \infty)$. The process $\{C(t), -\infty < t < \infty\}$ so obtained is a Markov chain in equilibrium with distribution given by (4). Moreover, the reversed process $\{C(-t), -\infty < t < \infty\}$ is also a Markov chain in equilibrium with the same distribution, the transition rates $q'(C)$, $q'(C, D)$ for the reversed process being given by (cf. [18] and [32])

$$\pi(C)\, q(C, D) = \pi(D)\, q'(D, C) \tag{11}$$

and

$$q'(C) = q(C) \ . \tag{12}$$

Now, using (11) and (12) it is not difficult to check (cf. [19]) that the reversed process is also an open Markovian network of queues. Observe that external arrivals for the reversed process correspond to external departures for the original process. Hence, external departures from node i in the original process are Poisson with rate $\lambda_i' = q_i \alpha_i$, and these external departure processes are independent for the various nodes. Also, since past departures from the system in the original process correspond to future external arrivals in the reversed process, the current state of the system in the original process is independent of past departures from the system.

2.3 Other Classical Work on Equilibrium Analysis

In contrast to the open network of queues introduced in Jackson (1957), Gordon and Newell (1967) considered a closed Markovian network in which a fixed and finite number of customers, say M, circulate through the network and no external arrivals or departures are permitted. This corresponds to Jackson's model in which $\lambda_i = 0$ and $q_i = 0$ for each i. As before, the state of the system is taken to be the numbers of customers at the various nodes. In this model, the state space $\mathscr{C}$ is the finite set of all N-tuples $C = (c_1, c_2, \ldots, c_N)$ of non-negative integers such that

$$c_1 + c_2 + \cdots + c_N = M \; . \tag{13}$$

The traffic equation now becomes

$$\alpha = \alpha P$$

and, assuming P is irreducible, we can regard α as the unique stationary distribution of the discrete parameter Markov chain with transition matrix P. Thus, for this model the process $\{C(t), t \geq 0\}$ is an irreducible continuous time Markov chain with finite state space, and hence possesses a limiting or equilibrium distribution. Let $\beta_i = \alpha_i/\mu_i$, let

$$\psi_i(c_i) = \begin{cases} \beta_i^{c_i}/c_i! & \text{if } c_i \leq s_i \text{, and} \\ \beta_i^{c_i}/s_i!\, s_i^{c_i - s_i} & \text{if } c_i > s_i \text{,} \end{cases} \tag{14}$$

for $i = 1, 2, \ldots, N$, and put

$$B^{-1} = \sum_{\mathscr{C}} \prod_{1}^{N} \psi_i(c_i) \quad . \tag{15}$$

Then, the result of Gordon and Newell is that $\pi(c_1, c_2, \ldots, c_N)$, the equilibrium probability of having c_1 customers at node 1 , c_2 customers at node 2 , and so on, is given by

$$\pi(c_1, c_2, \ldots, c_N) = B \prod_{1}^{N} \psi_i(c_i) \, . \tag{16}$$

Thus, the equilibrium distribution has a product form, though the behavior at the various nodes can no longer be regarded as independent since $(c_1, c_2 \ldots, c_N)$ must satisfy (13). Observe that in this closed system the quantity $\rho_i = \alpha_i/\mu_i s_i$ can be regarded as the "relative utilization" of node i . The result of Gordon and Newell can be derived using the technique of partial balance.

In [15] Jackson considered a more general open Markovian queueing system which also includes the model of Gordon and Newell as a special case. In the model of Jackson (1963) the customer arrival process is allowed to depend upon the total number of customers in the system and the service rate at any node is allowed to be a function of the number of customers at the node. In [15] Jackson provided sufficient conditions for the existence of a unique equilibrium distribution for the process $\{C(t), t \geq 0\}$ and showed that this equilibrium distribution is of the form (1).

In [36] the closed Markovian system of Gordon and Newell (1967) was generalized to allow different types of customers, with a different set of transition probabilities and service rates for each type. In addition, travel times between various nodes, instead of being taken as instantaneous, were permitted to have arbitrary distributions. Once again, it was demonstrated that the equilibrium distribution for the numbers of customers at the various nodes (and in transit) is of product form.

2.4 Limitations of Classical Work

The classical results on the equilibrium analysis of networks of queues have a number of limitations, including the following:

1. All service time distributions in the network are exponential.
2. For open systems, sufficient conditions are provided for the existence of equilibrium distributions, but these conditions are not shown to be necessary.
3. Customer-oriented measures of system performance such as total waiting time and total response time (waiting-plus-service time) experienced by an arbitrary customer have been largely ignored. The problems of response time and waiting time for a finite number of Markovian queues in tandem are extensively examined by Reich in [37], [38] and [39].
4. In the work of Jackson (1957,1963) and the work of Gordon and Newell (1967), all customer behavior is similar in the sense that at each node service times and transition probabilities are drawn from the same distribution and the queue discipline at each node is first-come-first served. Different customer

types are not permitted. Different customer types are allowed, however, in the closed system of Posner and Berhnoltz (1968).

5. A less serious (though not unimportant) limitation is the assumption of instantaneous travel times between nodes. This restriction is also removed for closed Markovian networks in [36].

As will be seen in the upcoming sections of this paper, recent efforts in networks of queues remove some of these limitations. But, we shall find that a number of important problems remain unsolved.

3.0 THE BASIC MODEL REVISITED

An interesting study of the classical queueing network model of Jackson (1957) <u>with single-server nodes</u> has recently been put forth in [3], [4], [30], and [31]. A noteworthy aspect of these papers is the emphasis placed on understanding the behavior and structure of the most fundamental model in the area of networks of queues. For purposes of information we remark that the results of [3], [4], and [31] are included in [30].

In this work a careful description of the network is provided and a careful formulation of the problem is presented. Open systems, closed systems, and mixed systems, i.e., systems which are neither open nor closed, are considered. An exhaustive study of the traffic equation is provided, followed by a thorough analysis of various aspects of system behavior under equilibrium conditions, including queue-length processes, departure processes and traffic processes on arcs. Also, the total service time received by an arbitrary customer is discussed. Some gaps in the extant theory for the model of Jackson (1957) are closed. In particular, for an open network with single-server nodes, the condition (3), i.e., $\rho_i < 1$ for each node i, is shown to be necessary for the existence of an equilibrium distribution (see [30]); also, as remarked earlier, the classical results for departure processes in M/M/s queues are extended to the network case for single-server nodes in [4]. Applications to equilibrium decompositions of the network are given. The results obtained are used to exhibit some structural simplifications that take "Jackson networks" into "Jackson networks," while preserving a variety of operating characteristics. Simulation complexities of "Jackson networks" are also discussed. Many of the results in [3], [4] and [30] are obtained by proceeding from the Kolomogoroff forward equation for an appropriate Markov jump process, while the results of [31], and many of the results in [30], are system-theoretic in nature.

For the basic model with single-server nodes, let T and S denote, respectively, the total time spent in the system and the total service time received by an arbitrary customer when the system is in equilibrium, that is, when the distribution of the queue-lengths vector $C(t)$ is given by (4) for all times t in $(-\infty, \infty)$. The Laplace transform of the distribution of S is derived in [3]. Combining the thrust of the argument for this service time result in [3] with a result in [4] concerning traffic processes on arcs, we now derive the Laplace transform of T <u>for the important case of an "acyclic network</u>," that is, an open network with "switching matrix" P such that a customer can never visit any node more than once. In the terminology of [4], each arc of the network is an "exit arc."

For an arbitrary customer entering the system at node i, let T_i denote the total time the customer is in the network. Let g_i denote the Laplace transform of the distribution of T_i for $i = 1, 2, \ldots, N$. For all t the queue-length distribution at node i coincides with that of an M/M/1 queue in equilibrium with traffic intensity ρ_i. So, a customer entering the system at node i remains at node i for a random time whose distribution has Laplace transform h_i where

$$h_i(u) = \frac{\nu_i}{\nu_i + u}$$

for $u \geq 0$ with $\nu_i = \mu_i(1 - \rho_i)$. After being served at node i, the customer either leaves the network (with probability q_i) or is transferred to some node $j \neq i$ (with probability p_{ij}). Conditioned on this latter event, the customer's total residual sojourn time in the network has the same

distribution as T_j and is independent of the time spent at i, the node of entry. To support this contention, we argue as follows. Let Q_i and Q_j, denote, respectively, the number of customers at node i and node j the instant after our customer has completed his service at node i. By Theorem 3.3 in [4], the traffic processes on the various arcs of the network are independent Poisson processes, the rate on arc (k, k') being $\alpha_k p_{kk'}$. Thus, the processes which measure the total stream of arrivals (both external and internal transfers) to the various nodes over finite time intervals are independent Poisson processes, the rate at node k being α_k. In view of this, the argument for Theorem 5 in [38] shows that the variables Q_i and Q_j are independent, conditioned on the transfer from i to j. Thus, our customer's sojourn time at node i is independent of the queue-size distribution at node j upon his arrival, and therefore independent of his total residual sojourn time in the network. Moreover, this total residual sojourn time has the same distribution as T_j. Therefore, we can write

$$g_i(u) = \left[q_i + \sum_j p_{ij} g_j(u)\right] \cdot h_i(u) \tag{17}$$

for $i = 1, 2, \ldots, N$. Substituting the above expression for h_i into (17) gives

$$[(\nu_i + u)/\nu_i] \cdot g_i(u) = q_i + \sum_j p_{ij} g_j(u) . \tag{18}$$

From here on we employ a simplified version of the arguments used in Section V of [3] to derive the Laplace transform of the distribution of S. Let $g(u)$

be a column vector whose ith component is $g_i(u)$, q of a column vector whose ith component is q_i, $D(u)$ a diagonal matrix whose ith diagonal entry is $(\nu_i + u)/\nu_i$, and let $G(u) = D(u) - P$. Then, equation (18) is equivalent to

$$G(u)\,g(u) = q . \tag{19}$$

The matrix $D(u)$ is clearly invertible for any $u \geq 0$, and so we can write

$$G(u) = \left(I - P[D(u)]^{-1}\right) D(u) . \tag{20}$$

Since each diagonal element of the diagonal matrix $[D(u)]^{-1}$ is in $(0, 1]$, it follows from (20) that $G(u)$ is invertible for any $u \geq 0$. An arbitrary customer enters the system via node i with probability $\lambda_i/(\lambda_1 + \lambda_2 + \ldots + \lambda_N)$. Let $\hat{\lambda}$ denote the row vector of these node-entry probabilities. Then, letting h denote the Laplace transform of the distribution of T, we therefore have

$$h(u) = \hat{\lambda}[G(u)]^{-1} q \tag{21}$$

for $u \geq 0$. It follows from (19) that each transform g_i is a rational function (whose denominator is a polynomial in u of degree at most N). Hence, $h(u)$ is the transform of a mixture of exponential distributions..

Having derived the Laplace transform of the distribution of T, let us now compute $E\{T\}$. Since any customer never visits a node more than once and each node functions as a $M/M/1$ queue in equilibrium, it is clear that $E\{T\}$ is finite. Thus

$$E\{T\} = -\lim_{u \to 0} h'(u)$$

where prime denotes derivative with respect to u . From (19) we have

$$g'(u) = -[G(u)]^{-1} G'(u) g(u) \tag{22}$$

for all $u > 0$. The matrix $G'(u)$ is simply the diagonal matrix whose ith diagonal entry is $1/\nu_i$, and as $u \to 0$ the vector $g(u)$ tends to the column vector of 1's . From (20) we have

$$[G(u)]^{-1} = \sum_{n=0}^{\infty} P^n \left([D(u)]^{-1}\right)^{n+1} \quad . \tag{23}$$

Since each element of the diagonal matrix $[D(u)]^{-1}$ is in $(0,1]$ and $[D(u)]^{-1}$ tends to the identity matrix as $u \to 0$, it follows readily from (23) that $[G(u)]^{-1}$ tends to $(I - P)^{-1}$ as $u \to 0$. Combining all of the above with the fact that $\alpha = \lambda(I - P)^{-1}$, we find that

$$E\{T\} = \frac{1}{\left(\sum_{i=1}^{N} \lambda_i\right)} \cdot \sum_{i=1}^{N} \frac{\rho_i}{1 - \rho_i} \quad . \tag{24}$$

For the case of a finite number of single-server Markovian queues in tandem, the results (21) and (24) reduce to those obtained by Reich in [38]. Replacing ν_i by μ_i in (18) yields the Laplace transform (and the first moment) of the distribution of S , the total amount of service received by an arbitrary

customer in an open network; see Section V of [3]. Thus, if we denote by W the total waiting time (exclusive of service time) experienced by an arbitrary customer, i.e., $W = T - S$, then for an open "acyclic network" we have

$$E\{W\} = \frac{1}{\left(\sum_{i=1}^{N} \lambda_i\right)} \cdot \sum_{i=1}^{N} \frac{\rho_i^2}{1 - \rho_i} . \tag{25}$$

3.1 Some Generalizations of Jackson (1957) and Gordon and Newell (1967)

In Kelly (1975) the classical Markovian queueing network models of Jackson (1957) and Gordon and Newell (1967) are generalized to networks of queues with different classes of server nodes and different types of customers while retaining the assumption that all service times are exponentially distributed and all external input processes are Poisson. In the model of Kelly (1975) each node has a single server, but by varying the service rate according to the number of customers in the node the node may function as a queue with one server, many servers (including an infinite number), or as a queue where the server works at a fixed rate but shares his effort equally among all customers present in the node (this being the "processor-sharing" discipline often appearing in the literature on computer time-sharing and multiprogramming systems). Also, when the maximum service rate is fixed, the allowable queue disciplines include first-come-first-served and preemptive-resume last-come-first-served. For both the closed system and the open system, equilibrium distributions are obtained for the numbers of customers at the various nodes, including

the arrangement of customer types within queues. The service time distributions are the same for all customers at a given node. These equilibrium distributions have a product form, generalizing what was observed for the classical models; and, for open systems, the various nodes are shown to be independent in equilibrium. The equilibrium distributions in Kelly (1975) are obtained using the method of partial balance. In addition, for the open system the departure processes in equilibrium of the various customer types from the different queues are shown to be independent Poisson streams; and, the present state of the process in equilibrium is shown to be independent of past departures from the system. These departure results are derived using time reversal.

The most general equilibrium results for extensions of the classical models of Jackson (1957) and Gordon and Newell (1967) have been obtained by Baskett et al (1975) and Reiser and Kobayashi (1975). These authors allow a variety of customer types and different kinds of server nodes in order to model central processors, data channels, terminals, and routing delays in computer systems. The allowable queue disciplines associated with the server nodes are first-come-first-served, processor sharing, no queueing (infinite number of servers), and preemptive-resume last-come-first-served. Each customer belongs to a single type while awaiting or receiving service, but may change type and node at the completion of a service request. State dependent arrival processes are allowed for open networks. The network may be closed for some customer types and open with respect to other customer types. If a node operates on a first-come-first-served basis, then the service time distribution is exponential and the same for all customer types, but the service rate may be state dependent.

At server nodes with other queue disciplines, service time distributions with rational Laplace transforms are permitted, with a different distribution for each type of customer. Using the method of partial balance, the equilibrium state probabilities, when they exist, are shown to have the general form (1).

In the particular case where the network is open and the arrival processes do not depend on the state of the system, the following interesting result is obtained in [2]. If the queue discipline at node i is first-come-first-served, or processor-sharing, or preemptive-resume last-come-first-served, then the equilibrium distribution of the total number of customers present at node i coincides with that of a $M/M/1$ system with an appropriate traffic intensity ρ_i. If node i has an infinite number of servers, then the equilibrium distribution of the total number of busy servers coincides with that of a $M/G/\infty$ system with an appropriate mean number of busy servers. Moreover, the various nodes are independent in equilibrium.

4.0 THE PAPERS OF KELLY (1976) AND BARBOUR (1976)

In Kelly (1976), the results of Jackson (1957, 1963) for open systems are expanded particularly with regard to routing behavior. Customers of different types are permitted along with a variety of routing schemes and queue disciplines. Customers of type m, $1 \leq m \leq M$, enter a system of queues i, $1 \leq i \leq N$, in a Poisson stream of rate ν_m, the separate streams being independent, and proceed through the fixed sequence of queues $r(m, 1), \ldots, r(m, L(m))$ before leaving the system. Hence, a customer of type m at stage ℓ $(1 \leq \ell \leq L(m))$ along his route will be at node $r(m, \ell)$. Each customer type is thus allowed to have a different route through the system. This permits, for example, a routing scheme whereby a customer visits each node exactly once, but in random order, before leaving the system; one can do this by having a separate customer type for each possible route. One can also have a routing scheme where each customer type visits some queue(s) exactly twice. More importantly, however, allowing fixed routes by customer type makes the model of Kelly (1976) well suited for communications applications. To continue with the model description, it is first assumed in [20] that at each node a customer requires an amount of service which is exponentially distributed. However, state-dependent service and arrival rates are allowed. The state of the system includes (for each node) the types of customers present and their positions in the queue, as well as the stage along his route that each customer has reached. The equilibrium state probabilities, when they exist, are shown to be of product form, with the state of each queue being independent of the rest of the system. Moreover, in equilibrium, customers of type m leave the system in a Poisson stream of rate ν_m, and the departure streams for the various customer types are independent.

In [20] the proof that the equilibrium state probabilities have a product form is accomplished not by appealing to the method of partial balance but by clever use of the reversed process, i.e., time reversal. To be more specific, let $\{C(t), 0 \leq t < \infty\}$ denote the Markov process describing the state of the system in [20] and suppose the transition rates of $\{C(t), 0 \leq t < \infty\}$ are $\{q(C), q(C, D)\}$. Further, suppose we believe the probability distribution $\{\pi(C)\}$ is the equilibrium distribution of $\{C(t), 0 \leq t < \infty\}$. This can be verified as follows. We first guess what the reversed process $\{C(-t), -\infty < t < \infty\}$ might be, that is to say, we guess the transition rates $\{q'(C), q'(C, D)\}$ for the reversed process. If

$$q'(C) = \sum_{D \neq C} q'(C, D)$$

for each possible network state C , and if $\{\pi(C)\}$, $\{q(C), q(C, D)\}$ and $\{q'(C), q'(C, D)\}$ together satisfy equations (11) and (12), then $\{C(t), 0 \leq t < \infty\}$ has equilibrium distribution $\{\pi(C)\}$ and the reversed process has transition rates $\{q'(C), q'(C, D)\}$; cf. [20]. For the particular model of Kelly (1976), this procedure of time reversal is much simpler than a direct appeal to equation (7) or to the technique of partial balance.

With the same formulation of customer types and customer routes by type, but under more restrictive assumptions on queue discipline, all of the above results in [20] are then generalized in [20] to the case where service times at the various nodes are allowed to be finite mixtures of gamma distributions. For these more general service time distributions, the permissible queue disciplines include last-come-first-served, no queueing, and processor-sharing,

but not first-come-first-served. The equilibrium state probabilities for this model have a product form, and the departure processes in equilibrium consist of independent Poisson streams. These results for more general service distributions are also established via time reversal as indicated above. It is then observed that the product form for the equilibrium distribution still obtains in the case of state-dependent arrival rates. Since the distribution of any positive random variable can be approximated arbitrarily closely by a finite mixture of gamma distributions, Kelly conjectures that his result holds for arbitrarily distributed service times. This conjecture is verified in Barbour (1976).

The particular model introduced by Kelly (1976) for which Barbour's result is obtained is as follows. As mentioned earlier, customers of several types enter a finite system of queues in independent Poisson streams, with customers of type m, say, passing through the fixed sequence of queues $r(m, 1), \ldots, r(m, L(m))$. Customers within each queue are ordered, and node i operates as follows:

(a) A customer of type m at stage ℓ along his route, with $r(m, \ell) = i$, requires a random amount of service $R_{m\ell}$ with finite mean.

(b) There is a single server at node i who supplies a total service effort at rate $\mu_i(c_i) > 0$ when c_i is the number of customers at node i.

(c) A proportion $\varphi_i(k, c_i)$ of this total service effort is directed to the customer in position k $(1 \leq k \leq c_i)$.

(d) A customer arriving at node i moves into position k with probability $\varphi_i(k, c_i + 1)$.

(e) The service requirements $(R_{m1}, \ldots, R_{mL(m)})$ of a type-m customer have a joint distribution F_m which is independent of the state of the network.

It should be noted that use of the same function φ_i in (c) and (d) permits such queue disciplines as processor-sharing, no queueing, and last-come-first-served, see [20], but <u>not</u> first-come-first-served.

Let $C(t) = [c_1(t), c_2(t), \ldots, c_N(t)]$ be the vector of queue-length processes at the various nodes at time $t > 0$. For $C = (c_1, c_2, \ldots, c_N)$, a possible state of the process $\{C(t), t \geq 0\}$, let

$$P(C) = \prod_{i=1}^{N} \left[A_i^{c_i} \Big/ \prod_{k=1}^{c_i} \mu_i(k) \right] , \tag{26}$$

where

$$A_i = \sum_{m=1}^{M} \nu_m \sum_{\ell=1}^{L(m)} 1_{\{r(m,s)=i\}} E\{R_{m\ell}\} \tag{27}$$

with $1_{\{\cdot\}}$ denoting indicator function. Given this setup, the essence of the result obtained by Barbour in [1] is as follows. Suppose that

$$\sum_{C} P(C) < \infty \tag{28}$$

and that for each m , $1 \leq m \leq M$, the joint distribution F_m has no mass at zero in any component, i.e., all service times for each customer type are strickly positive with probability one. Then, for each state C

$$\lim_{t \to \infty} P\{C(t) = C\} \propto P(C) . \tag{29}$$

Moreover, in equilibrium the various customer types depart the system in independent Poisson streams with rates $\nu_1, \ldots, \nu_M$. Barbour's result is achieved by proceeding from the results of Kelly (1976) and developing a sophisticated continuity argument involving weak convergence results.

5.0 SOME OPEN PROBLEMS

In this section we suggest some open problems in the equilibrium analysis of networks of queues. If one compares the limitations of the classical work cited in Section 2 with recent work reviewed in Sections 3 and 4, then it becomes obvious that while equilibrium queue distributions have been obtained for some very complex networks, there are significant gaps in what is known about important aspects of the equilibrium behavior of some quite fundamental models. Thus, there are important open problems which merit careful study, and we will now discuss some of these problems in separate numbered paragraphs.

1. Let us begin with the open network model of Jackson (1957) in which each node consists of a single server. Consider the random variable T , the total sojourn time in the system for an arbitrary customer, which was discussed in Section 2. In light of the equilibrium queue-lengths distribution (4), it is tempting to conjecture that the transform result (21) for an "acyclic network" also holds for a general Markovian open network. This impulse regarding (21) is tempered somewhat by results in [30] concerning the non-Poisson character of the total input process at a node i for which $p_{ii} > 0$. Nonetheless, it seems plausible that (21) should extend to a general open network. Reich observed in [39] that for two M/M/1 queues in tandem the waiting times (times spent in queue exclusive of service times) experienced by an arbitrary customer at each node were non-independent random variables. Thus, the distribution of the total waiting time in an open queueing network experienced by an arbitrary customer is probably difficult to analyze. For the basic model of Jackson (1957), there are several related problems posed in [30] concerning traffic processes on arcs.

2. It would be very interesting to have some equilibrium results for a network of M/G/1 queues, that is to say, a queueing network model of the sort introduced by Jackson (1957) with Poisson external input, single-server nodes, switching matrix P , a first-come-first served queue discipline, and arbitrarily distributed service times. More general service time distributions than the exponential are allowed in [2], as well as [1], but the results obtained are for queue disciplines other than first-come-first-served. One would certainly like to settle the question here of whether or not the equilibrium queue-lengths distribution has a product form, and, if so, whether the various nodes function as M/G/1 queues in equilibrium or, as in the open network model of [2], as M/M/1 queues in equilibrium.

3. Since many queueing systems encountered in practice operate just below the level of saturation, it would be helpful to have "heavy-traffic" results for networks of queues in the spirit of those obtained by Kingman (1965) and Köllerström (1974) for the GI/G/1 and GI/G/s queues. In an open network, for example, one could allow the traffic intensities at the various nodes to approach the value 1 from below by increasing the intensities of the input streams and then examine the asymptotic behavior of the equilibrium sojourn time distribution. Some results in this direction have been obtained in Harrison (1973) for a finite number of GI/G/1 queues in tandem. But much remains to be done.

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REFERENCES

[1] BARBOUR, A. D. (1976). Networks of Queues and the Method of Stages. Adv. Appl. Prob. 8, 584-591.

[2] BASKETT, F., CHANDY, K. M., MUNTZ, R. R. and PALACIOS, F. G. (1975). Open, Closed and Mixed Networks of Queues with Different Classes of Customers. J. Assoc. Comput. Mach. 22, 248-260.

[3] BEUTLER, F. J., MELAMED, B. and ZEIGLER, B. P. (1975). Equilibrium Properties of Arbitrarily Interconnected Queueing Networks. Technical Report 75-4, Computer, Information and Control Engineering Program, The University of Michigan. (See also Multivariate Analysis IV. (1976). P. Krishnaiah, Ed. North-Holland, Amsterdam.)

[4] BEUTLER, F. J. and MELAMED, B. (1975). Decomposition and Customer Streams of Feedback Queueing Networks in Equilibrium. Technical Report 75-5, Computer, Information and Control Engineering Program, The University of Michigan.

[5] BREIMAN, L. (1968). Probability. Addison-Wesley, Reading, Mass.

[6] CHUNG, K. L. (1967). Markov Chains with Stationary Transition Probabilities, 2nd Ed. Springer-Verlag, Berlin.

[7] ÇINLAR, E. (1975). An Introduction to Stochastic Processes. Prentice-Hall, Englewood Cliffs, N. J.

[8] DALEY, D. J. (1976). Queueing Output Processes. Adv. Appl. Prob. 8, 395-415.

[9] DISNEY, R. L. (1975). Random Flow in Queueing Networks: A Review and Critique. Trans. AIIE, 7, 268-288.

[10] GORDON, W. J. and NEWELL, G. F. (1967). Closed Queueing Systems with Exponential Servers. Operations Res. 15, 254-265.

[11] HARRISON, J. M. (1973). The Heavy Traffic Approximation for Single Server Queues in Series. J. Appl. Prob. 10, 613-629.

[12] IGLEHART, D. L. and LEMOINE, A. J. (1973). Approximations for the Repairman Problem with Two Repair Facilities, I: No Spares. Adv. Appl. Prob. 5, 595-613.

[13] IGLEHART, D. L. and LEMOINE, A. J. (1974). Approximations for the Repairman Problem with Two Repair Facilities, II: Spares. Adv. Appl. Prob. 6, 147-158.

[14] JACKSON, J. R. (1957). Networks of Waiting Lines. Operations Res. 5, 518-521.

[15] JACKSON, J. R. (1963). Jobshop-Like Queueing Systems. Management Sci. 10, 131-142.

[16] JACKSON, R. R. P. (1954). Queueing Systems with Phase-Type Service. Operational Res. Quart. 5, 109-120.

[17] JACKSON, R. R. P. (1956). Random Queueing Processes with Phase-Type Service. J. Roy. Statist. Soc. Ser. B. 18, 129-132.

[18] KARLIN, S. (1966). A First Course in Stochastic Processes. Academic Press, New York.

[19] KELLY, F. P. (1975). Networks of Queues with Customers of Different Classes. J. Appl. Prob. 12, 542-554.

[20] KELLY, F. P. (1976). Networks of Queues. Adv. Appl. Prob. 8, 416-432.

[21] KINGMAN, J. F. C. (1965). The Heavy Traffic Approximation in the Theory of Queues. Ref. [44], 137-169.

[22] KINGMAN, J. F. C. (1969). Markov Population Processes. J. Appl. Prob. 6, 1-18.

[23] KLEINROCK, L. (1975). Queueing Systems, Vol.1. Wiley-Interscience, New York.

[24] KLEINROCK, L. (1976). Queueing Systems, Vol.2. Wiley-Interscience, New York.

[25] KOBAYASHI, H. (1976). System Design and Performance Analysis Using Analytic Models. Software Modeling and Its Impact on Performance. K. M. Chandy and R. T. Yeh, Eds. Prentice-Hall, Englewood Cliffs, N. J.

[26] KOBAYASHI, H. and KONHEIM, A. G. (1977). Queueing Models for Computer-Communications System Analysis. IEEE Trans. Communications COM-25, 2-28.

[27] KOENIGSBERG, E. (1958) Cyclic Queues. Operational Res. Quart. 9, 22-35

[28] KÖLLERSTRÖM, J. (1974). Heavy Traffic Theory for Queues with Several Servers. I. J. Appl. Prob. 11, 544-552.

[29] LUREAU, F. (1974). A Queueing Theoretic Analysis of Logistics Repair Models with Spare Parts. Technical Report No. 55, Department of Operations Research, Stanford University.

[30] MELAMED, B. (1976). Analysis and Simplifications of Discrete Event Systems and Jackson Queueing Networks. Technical Report 76-6, Department of Industrial and Operations Engineering, The University of Michigan.

[31] MELAMED, B., ZEIGLER, B. P. and BEUTLER, F. J. (1975). Simplifications of Jackson Queueing Networks. Technical Report 163, Department of Computer and Communications Services, The University of Michigan.

[32] MORAN, P. A. P. (1968). An Introduction to Probability Theory. Clarendon Press, Oxford.

[33] OAKES, D. (1976). Random Overlapping Intervals - A Generalization of Erlang's Loss Formula. Ann. Probability, 4, 940-946.

[34] PITTEL, B. (1976). Closed Exponential Networks of Queues with Blocking: The Jackson-Type Stationary Distribution and Its Asymptotic Analysis. IBM Research Report No. 26548. IBM Thomas J. Watson Research Center, Yorktown Heights, New York.

[35] POLLARD, J. M. (1975). Combinatorial Properties of a Queueing System with Limited Availability. Adv. Appl. Prob. 7, 864-877.

[36] POSNER, M. and BERNHOLTZ, B. (1967). Closed Finite Queueing Networks with Time Lags and with Several Classes of Units. Operations Res. 16, 977-985.

[37] REICH, E. (1957). Waiting Times When Queues Are in Tandem. Ann. Math. Statist. 28, 768-773.

[38] REICH, E. (1963). Note on Queues in Tandem. Ann. Math. Statist. 34, 338-341.

[39] REICH, E. (1965). Departure Processes. Ref. [44], 439-457.

[40] REISER, M. and KOBAYASHI, H. (1975). Queueing Networks with Multiple Closed Chains: Theory and Computational Algorithms. IBM J. Res. Develop. 19, 283-294.

[41] REISER, M. and SAUER, C. H. (1976). Queueing Network Models: Methods of Solution and Their Program Implementation. IBM Research Report No. 26344, IBM Thomas J. Watson Research Center, Yorktown Heights, New York.

[42] REYNOLDS, J. F. (1975). The Covariance Structure of Queues and Related Processes - A Survey of Recent Work. Adv. Appl. Prob. 7, 383-415.

[43] SHANBAG, D. N. and TAMBOURATZIS, D. G. (1973). Erlang's Formula and Some Results on the Departure Process for a Loss System. J. Appl. Prob. 10, 233-240.

[44] SMITH, W. L. and WILKINSON, W. (1965). Proceedings of the Symposium on Congestion Theory. University of North Carolina Press, Chapel Hill.

[45] WHITTLE, P. (1967). Nonlinear Migration Processes. Proceedings of the 36th Session of the International Statistical Institute, 642-647.

[46] WHITTLE, P. (1968). Equilibrium Distributions for an Open Migration Process. J. Appl. Prob. 5, 567-571.

[47] WYSZEWIANSKI, R. J. and DISNEY, R. L. (1975). Feedback Queues in the Modeling of Computer Systems: A Survey. Technical Report No. 74-1, Department of Industrial and Operations Engineering, University of Michigan.